$P_1, \dots, P_s \in M$

$\mathcal{P}$ good polyhedral decomp

$Q_1, \dots, Q_s \in \mathbb{G}(\tilde{M})$ general

$q_i \not\in X_0 \qquad q_i = \overline{\mathbb{G}(L_i) \cdot Q_i} \cap X_0$

$h : (\Gamma, x_1, \dots, x_s) \longrightarrow M_{\mathbb{R}}$ marked tropical curve

simple genus 0

$h(x_i) = P_i$

$s = |\Delta| - 1$

Δ = degree of h

subdividing edges of Γ ⇝ new graph $\tilde{\Gamma}$ along with new

$h : \tilde{\Gamma} \longrightarrow M_{\mathbb{R}}$ can assume for $y \in \tilde{\Gamma}$, $h(y)$ vertex of $\mathcal{P}$ iff y is a vertex of $\tilde{\Gamma}$ or y is in marked unbounded edge.

($\tilde{\Gamma}$ could have bivalent vertices)

Can get another trop curve $h : \tilde{\Gamma} \to M_{\mathbb{R}}$

by removing all marked edges + any resulting bivalent vertices.
(identifying adjacent edges)

$\hat{\Gamma}^{[0]}$ = vertices in Γ not adjacent to marked edge

$\hat{\Gamma}$ has distinguished edges $E_1, -, E_s$ w/ the end point of E_{x_i} in the interior of E_i these are marked edges of $\hat{\Gamma}$.

Use this to do what? given tropical marked curve $h \rightsquigarrow$ count # of log curves $f^{\dagger} : (C^{\dagger}, x_i) \to X_0^{\dagger}$ through q_i
torically transverse

whose associated tropical curve is h $(\mathrm{Mult}(h))$

Before you do that you count the # of "pre-log" curve "things":

A <u>torically transverse prelog curve</u> in X_0 is a stable map

$f: \underset{\text{curve}}{C} \to X_0$ s.t. $\forall \underset{\text{vertex}}{v \in P}$ $f^{-1}(D_v) \to D_v$ is torically transverse curve.

and f satisfies the following:

(ab) If $p \in C$ s.t. $f(p) \in \operatorname{Sing}(X_0)$ then

① p is a double point of C  contained in two distinct irreducible components C_1, C_2 of C and $f(C_i) \subseteq D_{v_i}$ for distinct vertices $v_i \in P$ joined by an edge ω.

② let w_i = intersection multiplicity of C_i at p with $D_\omega \subseteq D_i$

(• that is if ϕ is regular function defined in a nbhd of $f(p)$ in D_{v_i} s.t. $(\phi = 0) = D_\omega$ (locally) then $f^*(\phi)$ regular function on C_i near p)

Then $w_1 = w_2$

A <u>line</u> on complete toric surface Y is a nonconstant torically transverse map $\phi: \mathbb{P}^1 \to Y$ s.t. $\#\phi^{-1}(\partial Y) \leqslant 3$ & $\#\phi^{-1}(D) \leqslant 1$ for any toric divisor on Y.

Prop 4.22

For each Edge $E \subseteq \hat{\Gamma}$ choose orientation

$\partial^- E$, ∂E^+ if bounded

$\partial^- E$ if unbounded

let $u_{(\partial^- E, E)} \in M$ primitive tangent vector to $h(E)$ pointing from $h(\partial^- E)$ to $h(\partial^+ E)$ or in unbounded direction of $h(E)$. Then the map:

$$\Phi : \mathrm{Map}(\hat{\Gamma}^{[0]}, M) \longrightarrow \prod_{E \in \hat{\Gamma}^{[1]} \setminus \hat{\Gamma}^{[1]}_\infty} M / \mathbb{Z} u_{(\partial^- E, E)} \times$$

these are what now.... ?

$$\times \prod_{i=1}^{s} M/\mathbb{Z}u_{(\partial^- E_i, E_i)}$$

$$H \longmapsto \Big(\big(H(\partial^+ E) - H(\partial^- E)\big)_E , \ \big(H(\partial^- E_i)\big)_i$$

Is an inclusion of lattices of finite index D

D = # of marked torically transverse pre-log curves up to isomorphism of form $f: (C, x_1, \ldots, x_i) \to X_0$

$f(x_i) = q_i$ and associated to tropical curve h.

using the two conditions we force a torically transverse pre-log curve to have we can cook up a tropical curve......

Log Curves

$f: C^\dagger \longrightarrow S^\dagger$ log smooth, (integral), relative dimension 1

<u>integral</u> is a condition on $\bar{M}_{S,f(x)} \longrightarrow \bar{M}_{C,x}$ won't say what it means but practically means <u>f</u> is flat.

Etale local description:

$S = \operatorname{Spec} A$ (A, m) strictly henselian local ring.
avil $k = \bar{k}$ or sep closed...
$0 \in S$ closed pt.

Choose chart:

$$Q = \bar{M}_{S,0} \xrightarrow{\sigma} A$$

that defines log structure on S.

$$\begin{tikzcd}
\bar{x} \in C_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\
0 \arrow[r] & S
\end{tikzcd}$$

étale locally around $\bar{x}$ we have: $C^{\dagger}$ is isomorphic to one of the following log schemes:

① $V = \operatorname{Spec} A[u]$

$$Q \to \mathcal{O}_V$$

$$q \mapsto f^{*}\sigma(q)$$

here $\underline{f}$ is smooth. No intrinsic new information given by log.

② $V = \operatorname{Spec} \dfrac{A[u,v]}{(uv-t)}$ some $t \in m_A$

$$\mathbb{N}^2 \oplus_{\mathbb{N}} Q \longrightarrow \mathcal{O}_V$$

$$((a,b), q) \longmapsto u^a v^b f^{*}\sigma(q)$$

relation

$$\begin{tikzcd} (1,1)=\alpha & & \\ \mathbb{N}^2 \oplus_{\mathbb{N}} Q \arrow[u, mapsto] & Q \arrow[l] & \alpha \\ \mathbb{N}^2 \arrow[u] & \mathbb{N} \arrow[l, "\Delta"'] \arrow[u] & 1 \arrow[u, mapsto] \end{tikzcd}$$

$\rightsquigarrow$ determined by f^{*} with $\sigma(\alpha) = t$

Here C_0 is nodal

③ $V = \operatorname{Spec} A[u]$

$$\mathbb{N} \oplus Q \to \mathcal{O}_V$$
$$(a, q) \mapsto u^a f^* \sigma(q)$$

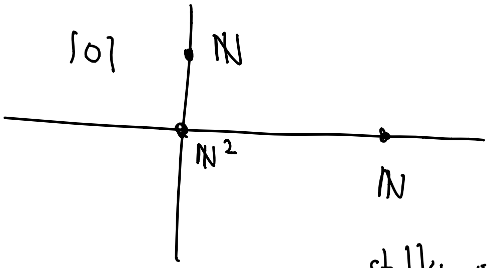
- Here $u=0$ is the image of a section $S \to C$ which we think of as a marked point.
- The log str. is the sum of that which is pulled back from the base + the divisorial log str coming from $(u=0)$.

Ex

$$\overline{M}_{(X,D),p} \hookrightarrow \mathbb{N}^{\#\text{ of components of } D \text{ passing through } p}$$

$X \downarrow$ smooth

$D \downarrow$ sncD



stalks of $\overline{M}_{(\mathbb{A}^2, V(xy))}$

$$\overline{M}_D := \iota_{X*}\mathbb{N} \oplus \iota_{Y*}\mathbb{N}$$

ι_{X}: inclusion of x axis into D

pullback of the above $\to$ log structure.

Given a torically pre-log curve $f: C \to X_0$ want to count # of non isomorphic log ~~structures~~ morphisms

$$\begin{tikzcd} f^{\dagger}: C^{\dagger} \arrow[r] \arrow[d] & X_0^{\dagger} \\ \operatorname{Spec} k^{\dagger} & \end{tikzcd}$$

with $\underline{f^{\dagger}} = \underline{f}$

↳ pulled back from X

- Have to impose some condition that f is strict where $X_0^{\dagger} \to \operatorname{Spec} k^{\dagger}$ is strict.

Recall $\partial X_0 = \overline{\partial X \setminus X_0} \cap X_0$

this is like the preimage of the boundary on X but the pullback log is a little more. than just that.

- Note $X_0^{\dagger} \to \operatorname{Spec} k^{\dagger}$ strict outside $\operatorname{Sing}(X_0) \cup \partial X_0$

$X_0 =$ Union of toric divisors D_v $v \in P$ vertex

∂X_0 like where X_0 "gently" meets others in ∂X

$\partial X = \bigcup D_{\text{rays in } \Sigma_P}$

$\overline{\mathcal{M}}_{(X,D),y} \hookrightarrow \mathbb{N}^{\#\text{ of components of } D \text{ passing through } y}$

$\Rightarrow$ for $p \in X_0 \setminus (\operatorname{Sing}(X_0) \cup \partial X_0)$ we have $\overline{\mathcal{M}}_{X_0,p} = \mathbb{N}$

these pts sit solidly in one component

$\mathcal{M}_{X_0,p} = \mathcal{O}^*_{X_0} \oplus \mathbb{N}$ which looks like $k^* \oplus \mathbb{N}$ pullbacked!

$\Rightarrow$ points on C which map to $X_0 \setminus (\operatorname{Sing}(X_0) \cup \partial X_0)$ must be smooth points of C.

- By the above double points (nodes) map into $\operatorname{Sing}(X_0)$
- It follows log markings must fall into ∂X_0. ⇝ only real consequence of structures.

Prop 4.23

Let $f: (C, x_1, \_, x_s) \to X_0$ be torically transverse pre-log curve constructed from $\underset{\text{simple}}{\wedge}$ h using Prop 4.22
tropical curve.

Assume further each edge $h(E)$ for $E \in \tilde{\Gamma}^{[1]} \setminus \tilde{\Gamma}^{[1]}_{\infty}$ has affine length divisible by $\omega(E)$ (this always happens after rescaling M) Then the # of nonisomorphic log maps

$$\begin{tikzcd} f^{\dagger}: (C^{\dagger}, x_1, \_, x_1) \arrow[r] \arrow[d] & X_0^{\dagger} \arrow[dl] \\ \operatorname{Spec} k^{\dagger} & \end{tikzcd}$$

with $\underline{f}^{\dagger} = f$ and strict where $X_0^{\dagger} \to \operatorname{Spec} k^{\dagger}$ is strict is

$$\left(\prod_{E \in \hat{\Gamma}^{[1]} \setminus \hat{\Gamma}^{[1]}_{\infty}} \omega(E) \right) \left(\prod_{i=1}^{s} \omega(E_i) \right)$$